On the consistency of an estimator for hierarchical Archimedean copulas

Jan Górecki[1], Marius Hofert[2], Martin Holeňa[3]

Abstract. The paper addresses an estimation procedure for hierarchical Archimedean copulas, which has been proposed in the literature. It is shown here that this estimation is not consistent in general. Furthermore, a correction is proposed, which leads to a consistent estimator.

Keywords: hierarchical Archimedean copula, Kendall distribution function, parameter estimation, structure determination, consistency

JEL classification: C51, C46
AMS classification: 62H99

1 Introduction

Hierarchical Archimedean copulas (HACs), which generalize Archimedean copulas (ACs) [13, p. 109], constitute a popular class of copulas, which has been used in high-dimensional applications. There exist successful applications of HACs in finance, e.g., for pricing collateralized debt obligations; see [4, 8]. Given some data, a HAC model can be build using several estimation techniques. Concerning those techniques, one can see that the estimation of a HAC can generally be divided in two subtasks: 1) structure determination and 2) parameter estimation. Both tasks are connected and hence, if the structured determination is performed poorly, i.e., if the true structure is not determined, then parameter estimation also leads to a poor result.

We are aware of two papers [2, 14] addressing both subtasks. The paper [14] describes a multi-stage HAC estimation procedure, in which the HAC structure is determined iteratively in a bottom-up manner. The estimation of the parameters is mainly performed using the maximum-likelihood (ML) technique, but the authors also briefly mention an alternative, which uses the relationship between the copula parameter and the value of Kendall's tau computed on a bivariate margin of the copula (shortly, $\theta - \tau$ relationship). The experimental results on simulated data show acceptable performance for the low-dimensional (5 variables) copula models considered in the paper. However, for some specific copula models, the quality of the results given by the mentioned procedure may be poor, as is addressed and experimentally shown in [2]. The paper [2], which empirically extends the theoretical procedure proposed in [1], also describes an alternative estimation procedure based on the $\theta - \tau$ relationship that can be successful in general.

Based on the research presented in the above-mentioned papers, we aim to show that, if the procedure presented in [14] is properly adjusted, the resulting estimator turns from a generally inconsistent estimator to a consistent estimator of the copula parameters.

The paper is structured as follows. Section 2 recalls some necessary theoretical concepts concerning copulas in general, ACs and HACs. Section 3 addresses the problem of the HAC estimation procedure described in [14]. Section 4 presents our proposed adjustment of the procedure and Section 5 concludes.

2 Preliminaries

2.1 Copulas

Definition 1. For every $d \geq 2$, a *d-dimensional copula* (shortly, *d-copula*) is a d-variate distribution

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function on $\mathbb{I}^d$, $\mathbb{I} = [0,1]$, whose univariate margins are uniformly distributed on $\mathbb{I}$.

Copulas establish a connection between a distribution function (df) H and its margins $F_1, ..., F_d$ (we use the term *margin* as an equivalent for *univariate margin*) via $H(\mathbf{x}) = C(F_1(x_1), ..., F_d(x_d))$, $\mathbf{x} \in \mathbb{R}^d$, as is well-known by Sklar's Theorem; see [15]. In case the margins $F_1, ..., F_d$ are all continuous, C is uniquely given by $C(u_1, ..., u_d) = H(F_1^-(u_1), ..., F_d^-(u_d))$, where $F_j^-, j \in \{1, ..., d\}$, denotes the pseudo-inverse of F_j given by $F_j^-(u) = \inf\{x \in \mathbb{R} \mid F_j(x) \geq u\},\ u \in \mathbb{I}$. As an example, elliptical copulas are derived in this way from multivariate elliptical dfs.

2.2 Archimedean Copulas

Archimedean copulas (ACs) are not constructed using Sklar's Theorem, but instead, one starts with a given functional form and asks for properties needed to obtain a proper copula. As a result of such a construction, ACs are expressed in closed form, which is one of the main advantages of this class of copulas. For construction of ACs, we need the following notions; see [12] for a more general construction of ACs.

Definition 2. An *Archimedean generator* (shortly, *generator*) is a continuous, nonincreasing function $\psi : [0, \infty] \to [0, 1]$, which satisfies $\psi(0) = 1,\ \psi(\infty) = \lim_{t\to\infty} \psi(t) = 0$ and is strictly decreasing on $[0, \inf\{t \in [0, \infty) \mid \psi(t) = 0\}]$. We denote the set of all such functions by Ψ.

Definition 3. A function f is called *completely monotone* (shortly, c.m.), if $(-1)^k f^{(k)}(x) \geq 0$ holds for every $k \in \mathbb{N}_0,\ x \in [0, \infty)$. We denote the set of all completely monotone generators by Ψ_∞.

Definition 4. Any d-copula C is called *Archimedean copula*, if it admits the form

$$C(\mathbf{u}) = C(\mathbf{u}; \psi) = \psi(\psi^{-1}(u_1) + ... + \psi^{-1}(u_d)),\ \mathbf{u} \in \mathbb{I}^d, \tag{1}$$

where $\psi \in \Psi$ and the $\psi^{-1} : [0, 1] \to [0, \infty]$ is defined $\psi^{-1}(s) = \inf\{t \in [0, \infty) \mid \psi(t) = s\},\ s \in \mathbb{I}$.

A condition sufficient for C to be a copula is stated as follows.

Theorem 1. *[10] If $\psi \in \Psi_\infty$, C given by (1) is a copula in any dimension d.*

2.3 Hierarchical Archimedean copulas

A copula is called *hierarchical Archimedean* if it is an AC with arguments possibly replaced by other hierarchical Archimedean copulas. In this paper, we consider the working example

$$C(\boldsymbol{u}) = C_0(u_1, C_1(u_2, u_3), C_2(u_4, u_5, C_3(u_6, u_7, u_8))), \tag{2}$$

where the *sector copula* C_k denotes an AC with generator ψ_k, $k \in \{0, 1, 2, 3\}$. Let us mention at this point that all presented ideas can be extended to arbitrary HACs (possibly at the cost of a significantly more complicated notation). Figure 1 shows the corresponding tree representation of this HAC.

According to *nesting condition* presented by [9, p. 88] and [11], nesting ACs leads to a proper copula if all nodes of the form $\psi_k^{-1} \circ \psi_l$ appearing in the hierarchical structure have completely monotone derivatives. In what follows, we assume this sufficient condition to hold. For a list of generators that fulfill this condition, see, e.g., [4, p. 65, pp. 115] or [5].

A random vector $\boldsymbol{U}$ distributed according to an HAC allows for a simple stochastic representation; see [6]. A random vector $\boldsymbol{U}$ distributed according to the HAC (2), e.g., can be represented as

$$\boldsymbol{U} = \left(\psi_0\Big(\frac{E_1}{V_0}\Big), \psi_1\Big(\frac{E_2}{V_{01}}\Big), \psi_1\Big(\frac{E_3}{V_{01}}\Big), \psi_2\Big(\frac{E_4}{V_{02}}\Big), \psi_2\Big(\frac{E_5}{V_{02}}\Big), \psi_3\Big(\frac{E_6}{V_{23}}\Big), \psi_3\Big(\frac{E_7}{V_{23}}\Big), \psi_3\Big(\frac{E_8}{V_{23}}\Big)\right), \tag{3}$$

where $E_j,\ j \in \{1, \ldots, 8\}$, are i.i.d. random variables with standard exponential distribution Exp(1), independent of the random variables V_0, V_{01}, V_{02} and V_{23}. The most important ingredients of the stochastic representation (3) are the random variables V_0, V_{01}, V_{02} and V_{23}. The dependence among these random variables determines the tree structure of the HAC. It can be described as follows. First, draw $V_0 \sim F_0(x) = \mathcal{LS}^{-1}[\psi_0](x)$, where $\mathcal{LS}$ denotes Laplace-Stieltjes transform, i.e., V_0 is distributed according to the distribution whose Laplace-Stieltjes transform is ψ_0. Next, draw $V_{01}|V_0 \sim F_{01}(x; V_0) =$

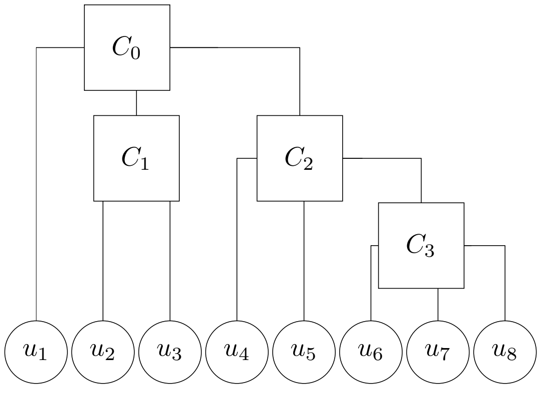

Figure 1 Tree structure of the HAC C as given in (2).

$\mathcal{LS}^{-1}[\psi_{01}(\cdot\,;V_0)](x)$, where $\psi_{01}(t;V_0) = \exp\big(-V_0\psi_0^{-1}(\psi_1(t))\big)$. Similarly, draw $V_{02}|V_0 \sim F_{02}(x;V_0) = \mathcal{LS}^{-1}[\psi_{02}(\cdot\,;V_0)](x)$, where $\psi_{02}(t;V_0) = \exp\big(-V_0\psi_0^{-1}(\psi_2(t))\big)$. Finally, on the innermost nesting level of the copula C in (2), the random variable V_{23} resides. Its distribution is determined by $V_{23}|V_{02} \sim F_{23}(x;V_{02}) = \mathcal{LS}^{-1}[\psi_{23}(\cdot\,;V_{02})](x)$, where $\psi_{23}(t;V_{02}) = \exp\big(-V_{02}\psi_2^{-1}(\psi_3(t))\big)$. To summarize, the tree-like (or cascading) dependence structure of a HAC stems from the distribution of the random variables V_0, V_{01}, V_{02} and V_{23}. Furthermore, note that the HAC (2) is simply survival copula [4, p. 36] of the random vector

$$\left(\frac{E_1}{V_0}, \frac{E_2}{V_{01}}, \frac{E_3}{V_{01}}, \frac{E_4}{V_{02}}, \frac{E_5}{V_{02}}, \frac{E_6}{V_{23}}, \frac{E_7}{V_{23}}, \frac{E_8}{V_{23}}\right).$$

Indeed, E_1/V_0 has survival function ψ_0, E_j/V_{01} has survival function ψ_1, $j \in \{2,3\}$, E_j/V_{02} has survival function ψ_2, $j \in \{4,5\}$, and E_j/V_{23} has survival function ψ_3, $j \in \{6,7,8\}$. To check the last, e.g., note that

$$\begin{aligned}
\mathbb{P}(E_j/V_{23} > t) &= \mathbb{P}(E_j > tV_{23}) = \mathbb{E}[\mathbf{1}_{E_j > tV_{23}}] = \mathbb{E}[\mathbb{E}[\mathbf{1}_{E_j > tV_{23}}|V_{23}]] = \mathbb{E}[\exp(-V_{23}t)] \\
&= \mathbb{E}[\mathbb{E}[\exp(-V_{23}t)|V_{02}]] = \mathbb{E}\big[\exp\big(-V_{02}\psi_2^{-1}(\psi_3(t))\big)\big] \\
&= \mathbb{E}\big[\mathbb{E}\big[\exp\big(-V_{02}\psi_2^{-1}(\psi_3(t))\big)|V_0\big]\big] = \mathbb{E}\Big[\exp\Big(-V_0\psi_0^{-1}\big(\psi_2\big(\psi_2^{-1}(\psi_3(t))\big)\big)\Big)\Big] \\
&= \mathbb{E}\big[\exp\big(-V_0\psi_0^{-1}(\psi_3(t))\big)\big] = \psi_0\big(\psi_0^{-1}(\psi_3(t))\big) = \psi_3(t).
\end{aligned}$$

3 Transformation using the Kendall distribution function

First, consider the sector copula C_3 and note that $(U_6, U_7, U_8) \sim C_3$. After having estimated the parameter(s) of C_3 (see [7] for some available AC estimators), one can let the vector (U_6, U_7, U_8) collapse to a single component $U_{K_{3,3}}$ in such a way that the parameters of the dependence structure of $(U_1, \dots, U_5, U_{K_{3,3}})$ can be estimated. Due to the popularity of the Kendall distribution function, one might be tempted to choose $U_{K_{3,3}}$ as $K_{3,3}\big(\psi_3((E_6+E_7+E_8)/V_{23})\big)$, where $K_{k,l}$ denotes the Kendall distribution function in k dimensions based on ψ_l and the term $\psi_3((E_6+E_7+E_8)/V_{23})$ is the Kendall transformation of the vector (U_6, U_7, U_8), i.e., $\psi_3((E_6+E_7+E_8)/V_{23}) = \psi_3(\psi_3^{-1}(\psi_3(E_6/V_{23}))+\psi_3^{-1}(\psi_3(E_7/V_{23}))+\psi_3^{-1}(\psi_3(E_8/V_{23}))) = C_3(U_6, U_7, U_8)$. This approach is suggested by [14]. We will now show that $(U_1, \dots, U_5, U_{K_{3,3}})$ does in general *not* have the copula

$$C_0\big(u_1, C_1(u_2, u_3), C_2(u_4, u_5, u_6)\big) \tag{4}$$

as distribution function. It follows that this estimation procedure is inconsistent for the parameters that correspond to all but the innermost sector copulas. This fact is supported empirically by the simulation studies in [2, 14] and has also been suspected in [3]. In what follows, we derive the copula of $(U_1, \dots, U_5, U_{K_{3,3}})$. Furthermore, we suggest an approach how to collapse (U_6, U_7, U_8) in such a way that we obtain a random vector following (4).

For $u_1, ..., u_6 \in [0,1]$, assume $x_1 = \psi_0^{-1}(u_1), x_2 = \psi_1^{-1}(u_2), x_3 = \psi_1^{-1}(u_3), x_4 = \psi_2^{-1}(u_4), x_5 = \psi_2^{-1}(u_5)$

and $x_6 = \psi_3^{-1}(K_{3,3}^{-1}(u_6))$. By stepwise conditioning on V_0, V_{01}, V_{02} and V_{23}, one can find that

$$\begin{aligned}
&\mathbb{P}\left(\frac{E_1}{V_0} > x_1, \frac{E_2}{V_{01}} > x_2, \frac{E_3}{V_{01}} > x_3, \frac{E_4}{V_{02}} > x_4, \frac{E_5}{V_{02}} > x_5, \frac{E_6+E_7+E_8}{V_{23}} > x_6\right) \\
&= \int_0^\infty \exp(-v_0 x_1)\left(\int_0^\infty \exp(-v_{01}(x_2+x_3))\, dF_{01}(v_{01}; v_0) \int_0^\infty \exp(-v_{02}(x_4+x_5))\right. \\
&\cdot \left.\int_0^\infty \mathbb{P}(E_6+E_7+E_8 > V_{23}x_6 | V_{23} = v_{23})\, dF_{23}(v_{23}; v_{02}) dF_{02}(v_{02}; v_0)\right) dF_0(v_0). \qquad (5)
\end{aligned}$$

Taking into account the fact that the sum of n independent Exp(1) distributed random variables follows an Erlang distribution with survival function $\bar{F}_{Erl,n}(x) = \exp(-x)\sum_{k=0}^{n-1} x^k/k!$, $x \in [0,\infty)$, we obtain

$$\begin{aligned}
&\int_0^\infty \mathbb{P}(E_6+E_7+E_8 > V_{23}x_6 | V_{23} = v_{23})\, dF_{23}(v_{23}; v_{02}) = \int_0^\infty \mathbb{P}(E_6+E_7+E_8 > v_{23}x_6)\, dF_{23}(v_{23}; v_{02}) \\
&= \sum_{k=0}^{2} \frac{(-x_6)^k}{k!} \int_0^\infty (-v_{23})^k \exp(-v_{23}x_6)\, dF_{23}(v_{23}; v_{02}) = \sum_{k=0}^{2} \frac{(-x_6)^k}{k!} \int_0^\infty \frac{d^k}{dx_6^k} \exp(-v_{23}x_6)\, dF_{23}(v_{23}; v_{02}) \\
&= \sum_{k=0}^{2} \frac{(-x_6)^k}{k!} \frac{d^k}{dx_6^k} \mathbb{E}[\exp(-V_{23}x_6) | V_{02} = v_{02}].
\end{aligned}$$

Using linearity, (5) therefore equals

$$\begin{aligned}
&\sum_{k=0}^{2} \frac{(-x_6)^k}{k!} \frac{d^k}{dx_6^k} \int_0^\infty \exp(-v_0 x_1)\left(\int_0^\infty \exp(-v_{01}(x_2+x_3))\, dF_{01}(v_{01}; v_0)\right. \\
&\quad \cdot \left.\int_0^\infty \exp(-v_{02}(x_4+x_5)) \mathbb{E}[\exp(-V_{23}x_6) | V_{02} = v_{02}]\, dF_{02}(v_{02}; v_0)\right) dF_0(v_0) \\
&= \sum_{k=0}^{2} \frac{(-x_6)^k}{k!} \frac{d^k}{dx_6^k} \mathbb{E}\Big[\exp(-V_0 x_1) \mathbb{E}[\exp(-V_{01}(x_2+x_3)) | V_0] \\
&\qquad \cdot \mathbb{E}\Big[\exp(-V_{02}(x_4+x_5)) \mathbb{E}[\exp(-V_{23}x_6) | V_{02}] \Big| V_0\Big]\Big] \\
&= \sum_{k=0}^{2} \frac{(-x_6)^k}{k!} \frac{d^k}{dx_6^k} \mathbb{E}\Big[\exp(-V_0 x_1) \mathbb{E}[\exp(-V_{01}(x_2+x_3)) | V_0] \\
&\qquad \cdot \mathbb{E}\Big[\exp(-V_{02}(x_4+x_5)) \exp\big(-V_{02}\psi_2^{-1}(\psi_3(x_6))\big) \Big| V_0\Big]\Big] \\
&= \sum_{k=0}^{2} \frac{(-x_6)^k}{k!} \frac{d^k}{dx_6^k} \mathbb{E}\Big[\exp(-V_0 x_1) \exp\big(-V_0 \psi_0^{-1}(\psi_1(x_2+x_3))\big) \\
&\qquad \cdot \exp\Big(-V_0 \psi_0^{-1}\big(\psi_2\big(x_4+x_5+\psi_2^{-1}(\psi_3(x_6))\big)\big)\Big)\Big] \\
&= \sum_{k=0}^{2} \frac{(-x_6)^k}{k!} \frac{d^k}{dx_6^k} \psi_0\Big(x_1 + \psi_0^{-1}(\psi_1(x_2+x_3)) + \psi_0^{-1}\big(\psi_2\big(x_4+x_5+\psi_2^{-1}(\psi_3(x_6))\big)\big)\Big),
\end{aligned}$$

so that the copula of $(U_1, \dots, U_5, U_{K_{3,3}})$ at $(u_1, \dots, u_6)$ is not (4) but

$$\sum_{k=0}^{2} \frac{\big(-\psi_3^{-1}(K_{3,3}^{-1}(u_6))\big)^k}{k!} \frac{d^k}{dx_6^k} C_0\big(u_1, C_1(u_2,u_3), C_2(u_4,u_5,\psi_3(x_6))\big)\Big|_{x_6 = \psi_3^{-1}(K_{3,3}^{-1}(u_6))}. \qquad (6)$$

Although the implications of the representation (6) are unclear at the moment, we can interpret (6) as the Taylor polynomial of order two of $x_6 \mapsto C_0\big(u_1, C_1(u_2,u_3), C_2(u_4,u_5,\psi_3(x_6))\big)$ about $\psi_3^{-1}(K_{3,3}^{-1}(u_6))$ evaluated at zero. If the dimension of the sector copula C_3 converges to infinity, we get (with the same argument) the Taylor polynomial of order infinity and hence the function itself evaluated at 0, which is $C_0(u_1, C_1(u_2,u_3), C_2(u_4,u_5,\psi_3(x_6)))$ at $x_6 = 0$ and thus $C_0(u_1, C_1(u_2,u_3), C_2(u_4,u_5))$, which is the marginal copula of C for $u_6 = u_7 = u_8 = 1$.

4 Transformation using the diagonal of the Archimedean copula

As becomes directly clear from (2) or its tree representation in Figure 1, if we build marginal copulas of (2) for two of the components with index in $\{6, 7, 8\}$ being 1, we obtain (4). This means that after estimation of the parameters of C_2 one could (e.g., randomly) choose one of the components with index in $\{6, 7, 8\}$ for continuing with the parameter estimation on higher levels in (4). On the one hand, an advantage would be that the estimation error for the parameter of C_3 is not inherited. On the other hand, choosing just one component means reducing the available information. In what follows, we therefore describe a complementary procedure that is based on the copula diagonal of C_3 and shares the properties that are opposite to those mentioned above.

For constructing a random vector distributed according to (4), one can consider

$$U_{\delta_{3,3}} = \delta_{3,3}(\max\{U_6, U_7, U_8\}), \tag{7}$$

where $\delta_{k,l}(t) := \psi_l(k\psi_l^{-1}(t))$ denotes the diagonal in dimension k of the AC generated by ψ_l [7]. Note that $\mathbb{P}(U_{\delta_{3,3}} \leq x) = \mathbb{P}(\max\{U_6, U_7, U_8\} \leq \psi_3(\psi_3^{-1}(x)/3)) = \mathbb{P}(U_6 \leq \psi_3(\psi_3^{-1}(x)/3)), U_7 \leq \psi_3(\psi_3^{-1}(x)/3)), U_8 \leq \psi_3(\psi_3^{-1}(x)/3))) = C_3(\psi_3(\psi_3^{-1}(x)/3),\ \psi_3(\psi_3^{-1}(x)/3), \psi_3(\psi_3^{-1}(x)/3)) = x$, provided C_3 is an AC with generator ψ_3, hence $U_{\delta_{3,3}} \sim \mathrm{U}[0, 1]$, where $\mathrm{U}[0, 1]$ denotes the univariate uniform distribution on $[0, 1]$. Since the maximum is distributed according to the copula diagonal, see that $\delta_{3,3}^{-1}(U) \sim \delta_{3,3}$ for any $U \sim \mathrm{U}[0, 1]$, one effectively replaces the copula C_3 by its diagonal. Assume $x_1, ..., x_5$ as above and $x_6 = \psi_3^{-1}(u_6)$, and note that $\psi_3^{-1}(U_{\delta_{3,3}}) = 3\big(\psi_3^{-1}\big(\max\{\psi_3(\frac{E_6}{V_{23}}), \psi_3(\frac{E_7}{V_{23}}), \psi_3(\frac{E_8}{V_{23}})\}\big)\big) = 3\min\{\frac{E_6}{V_{23}}, \frac{E_7}{V_{23}}, \frac{E_8}{V_{23}}\}$. Then the resulting survival function can be computed as

$$\begin{aligned}
&\mathbb{P}\Big(\frac{E_1}{V_0} > x_1, \frac{E_2}{V_{01}} > x_2, \frac{E_3}{V_{01}} > x_3, \frac{E_4}{V_{02}} > x_4, \frac{E_5}{V_{02}} > x_5, \frac{3\min\{E_6, E_7, E_8\}}{V_{23}} > x_6\Big)\\
&= \int_0^\infty \exp(-v_0 x_1)\Big(\int_0^\infty \exp(-v_{01}(x_2 + x_3))\, dF_{01}(v_{01}; v_0) \int_0^\infty \exp(-v_{02}(x_4 + x_5))\\
&\qquad \cdot \int_0^\infty \exp\Big(-v_{23}\Big(\frac{x_6}{3} + \frac{x_6}{3} + \frac{x_6}{3}\Big)\Big)\, dF_{23}(v_{23}; v_{02})\, dF_{02}(v_{02}; v_0)\Big)\, dF_0(v_0)\\
&= \mathbb{E}\Big[\exp(-V_0 x_1)\mathbb{E}[\exp(-V_{01}(x_2 + x_3))|V_0]\\
&\qquad \cdot \mathbb{E}\Big[\exp(-V_{02}(x_4 + x_5))\mathbb{E}\Big[\exp\Big(-V_{23}\Big(\frac{x_6}{3} + \frac{x_6}{3} + \frac{x_6}{3}\Big)\Big)\Big|V_{02}\Big]\Big|V_0\Big]\Big]\\
&= \mathbb{E}\Big[\exp(-V_0 x_1)\mathbb{E}[\exp(-V_{01}(x_2 + x_3))|V_0]\mathbb{E}\big[\exp(-V_{02}(x_4 + x_5))\exp\big(-V_{02}\psi_2^{-1}(\psi_3(x_6))\big)\big|V_0\big]\Big]\\
&= \mathbb{E}\Big[\exp(-V_0 x_1)\mathbb{E}[\exp(-V_{01}(x_2 + x_3))|V_0]\mathbb{E}\Big[\exp\big(-V_{02}\big(x_4 + x_5 + \psi_2^{-1}(\psi_3(x_6))\big)\big)\Big|V_0\Big]\Big]\\
&= \mathbb{E}\Big[\exp(-V_0 x_1)\exp\big(-V_0\psi_0^{-1}(\psi_1(x_2 + x_3))\big)\exp\Big(-V_0\psi_0^{-1}\big(\psi_2\big(x_4 + x_5 + \psi_2^{-1}(\psi_3(x_6))\big)\big)\Big)\Big]\\
&= \mathbb{E}\Big[\exp\Big(-V_0\Big(x_1 + \psi_0^{-1}(\psi_1(x_2 + x_3)) + \psi_0^{-1}\big(\psi_2\big(x_4 + x_5 + \psi_2^{-1}(\psi_3(x_6))\big)\big)\Big)\Big)\Big]\\
&= \psi_0\Big(x_1 + \psi_0^{-1}(\psi_1(x_2 + x_3)) + \psi_0^{-1}\big(\psi_2\big(x_4 + x_5 + \psi_2^{-1}(\psi_3(x_6))\big)\big)\Big),
\end{aligned}$$

so that the copula of $(U_1, \ldots, U_5, U_{\delta_{3,3}})$ at $(u_1, \ldots, u_6)$ is indeed the HAC (4). We can therefore continue in the same fashion with estimating and further collapsing the sectors on the innermost nesting levels of this copula, i.e, in the same fashion as in the multi-stage procedure described in [14].

5 Conclusion

To summarize, when using the transformation based on the Kendall distribution function, the copula of $(U_4, U_5, U_{K_{3,3}})$ is generally not C_2. Hence, if we wrongly assume that $(U_4, U_5, U_{K_{3,3}}) \sim C_2$, then even if a consistent estimator for the parameters of the copula C_2 is used, the resulting estimator is in general inconsistent for the parameter(s) of C_2. In contrast, if the transformation based on the diagonal of the AC is used (or any component with index in $\{6, 7, 8\}$), it is assured that the sector copulas in higher nesting levels of the estimated HAC are the distribution functions of the corresponding random vectors, i.e., $(U_4, U_5, U_{\delta_{3,3}}) \sim C_2$ and $(U_1, U_{\delta_{2,1}}, U_{\delta_{3,2}}) \sim C_0$. Using any consistent estimator for the parameters of an AC, the resulting estimator for the parameters of the HAC is consistent.

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