An Alternative Approach to the Structure Determination of Hierarchical Archimedean Copulas

Jan Górecki[1], Martin Holeňa[2]

Abstract. Copulas offer a flexible tool for a stochastic dependence modeling. One of the most popular classes of copulas is the class of hierarchical Archimedean copulas, which gained its popularity due to the fact that the models from the class are able to model the stochastic dependencies conveniently even in high dimensions. One critical issue when estimating a hierarchical Archimedean copula is to correctly determine its structure. The paper describes an approach to the problem of the structure determination of a hierarchical Archimedean copula, which is based on the close relationship of the copula structure and the values of measure of concordance computed on all its bivariate margins. The presented approach is conveniently summarized as a simple algorithm.

Keywords: hierarchical Archimedean copula, structure determination, measure of concordance, bivariate margins, nesting condition

JEL classification: C51, C46
AMS classification: 62H99

1 Introduction

Hierarchical Archimedean copulas (HACs), which generalize Archimedean copulas (ACs), overcome some limitations and bring some advantages compared to the most popular class of Gaussian copulas [2]. There already emerged successful applications of HACs in finance, e.g., in collateral debt obligation pricing, see [2, 5]. One critical issue when estimating HAC is to properly determine its structure. Despite the popularity of HACs, there exists only one paper [9] addressing generally the structure determination. The method presented in that paper mainly focus on maximum likelihood estimation (MLE) for the estimation of HAC's parameters, which are later used for the structure determination. The MLE used in the method involves the computation of the density of a HAC that needs up to d derivatives, where d is the data dimension. The authors claim, that the approach is feasible in high dimensions when using numerical method for the density computation and present two examples for $d = 5$, which involves only homogeneous HAC and which incorporates ACs belonging to one Archimedean family. Our approach provides an alternative way to the problem, which completely avoids the need of the HAC's density computation for some Archimedean families, hence is feasible even in very high dimensions.

The paper is structured as follows. The second section recalls some necessary theoretical concepts concerning copulas, the third section presents the proposed approach to the structure determination of HAC and the fourth section concludes the paper.

2 Preliminaries

2.1 Copulas

Definition 1. For every $d \geq 2$, a *d-dimensional copula* (shortly, *d-copula*) is a d-variate distribution function on $\mathbb{I}^d$ ($\mathbb{I}$ is unit interval), whose univariate margins are uniformly distributed on $\mathbb{I}$.

At the first look, copulas (denote the set of all copulas as $\mathcal{C}$) form one of many classes of joint distribution functions (shortly, joint d.f.s). What makes copulas interesting is that they establish a

[1]SBA in Karvina, Silesian University in Opava, Karvina, Czech Republic, Department of informatics, gorecki@opf.slu.cz
[2]Institute of Computer Science, Academy of Sciences of the Czech Republic, Praha, Czech Republic, martin@cs.cas.cz

connection between general joint d.f. and its univariate margins (in text below we use only *margin* for term *univariate margin*).

Theorem 1. ***(Sklar's Theorem)*** *[10] Let H be a d-dimensional d.f. with margins $F_1, ..., F_d$. Let A_j denote the range of F_j, $A_j := F_j(\overline{\mathbb{R}})(j = 1, ..., d), \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Than there exists a copula C such for all $(x_1, ..., x_d) \in \overline{\mathbb{R}}^d$,*

$$H(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)). \tag{1}$$

Such a C is uniquely determined on $A_1 \times ... \times A_d$ and, hence, it is unique if $F_1, ..., F_d$ are all continuous.

Through the Sklar's theorem, one can derive for any d-variate d.f. its copula C using (1). In case that the margins $F_1, ..., F_d$ are all continuous, copula C is given by $C(u_1, ..., u_d) = H(F_1^-(u_1), ..., F_d^-(u_d))$, where $F_i^-, i \in \{1, ..., d\}$ denotes pseudo-inverse of F_i given by $F_i^-(s) = \inf\{t|\ F_i(t) \geq s\}, s \in \mathbb{I}$. Many classes of copulas are derived in this way from popular joint d.f.s, e.g., the most popular class of Gaussian copulas is derived using H corresponding to d-variate Gaussian distribution. But, using this process often results in copula forms not representable in closed form, what can bring difficulties in some applications.

2.2 Archimedean Copulas

This drawback is overcame while using Archimedean copulas due to their different construction process. ACs are not constructed using thr Sklar's theorem, but instead of it, one starts with a given functional form and asks for properties in order to obtain a proper copula. As a result of such a construction, ACs are always expressed in closed form, which is one of the main advantages of this class of copulas [3]. To construct ACs we need a notion of an *Archimedean generator* and a *complete monotonicity.*

Definition 2. *Archimedean generator* (shortly, *generator*) is continuous, nonincreasing function $\psi : [0, \infty] \to [0, 1]$, which satisfies $\psi(0) = 1, \psi(\infty) = \lim_{t\to\infty} \psi(t) = 0$ and is strictly decreasing on $[0, \inf\{t : \psi(t) = 0\}]$.

Remark 1. We denote set of all generators as Ψ.

Definition 3. Function f is called *completely monotone* (shortly, c.m.) on $[a, b]$, if $(-1)^k f^{(k)}(x) \geq 0$ holds for every $k \in \mathbb{N}_0, x \in (a, b)$.

Definition 4. Any d-copula C is called *Archimedean copula* (we denote it d-AC), if it admits the form

$$C(\mathbf{u}) := C(\mathbf{u}; \psi) := \psi(\psi^{-1}(u_1) + ... + \psi^{-1}(u_d)), \mathbf{u} \in \mathbb{I}^d, \tag{2}$$

where $\psi \in \Psi$ and its inverse $\psi^{-1} : [0, 1] \to [0, \infty]$ is defined $\psi^{-1}(0) = \inf\{t : \psi(t) = 0\}$.

For verifying whether function C given by (2) is a proper copula, we can use the property stated in Definiton 3. A condition sufficient[1] for C to be a copula is stated as follows.

Theorem 2. *If $\psi \in \Psi$ is completely monotone, then function C given by (2) is copula.*

We can see from Definition 4 that having a random vector $\mathbf{U}$ distributed according to some AC, all its k-dimensional ($k < d$) marginal copulas have the same marginal distribution. It implies that all multivariate margins of the same dimension are equal, thus, e.g., the dependence among all pairs of components is identical. This symmetry of ACs is often considered to be a rather strong restriction, especially in high dimensional applications.

2.3 Hierarchical Archimedean Copulas

To allow for asymmetries, one may consider the class of HACs[2], recursively defined as follows.

Definition 5. A d-dimensional copula C is called *hierarchical Archimedean copula* if it is an AC with arguments possibly replaced by other hierarchical Archimedean copulas. If C is given recursively by (2) for $d = 2$ and

$$C(\mathbf{u}; \psi_0, ..., \psi_{d-2}) = \psi_0(\psi_0^{-1}(u_1) + \psi_0^{-1}(C(u_2, ..., u_d; \psi_1, ..., \psi_{d-2}))), \mathbf{u} \in \mathbb{I}^d, \tag{3}$$

[1] Necessary and sufficient condition for C to be a copula can be found in [6]

[2] often also called *nested Archimedean copulas*

for $d \geq 3$, C is called *fully-nested hierarchical Archimedean copula* with $d-1$ *nesting levels*. Otherwise C is called *partilally-nested hierarchical Archimedean copula*. [4]

Remark 2. We denote a d-dimensional HAC as d-HAC. We refer to the hierarchical ordering of $C(\cdot;\psi_0), \ldots, C(\cdot;\psi_{d-2})$ together with the ordering of variables $u_1, \ldots, u_d$ as the *structure* of a d-HAC.

From the definition, we can see that ACs are special cases of HACs. The most simple proper fully-nested HAC is copula C obtained for $d = 3$ with two nesting levels. The structure of this copula is given by

$$\begin{aligned} C(\mathbf{u};\psi_0,\psi_1) &= C(u_1, C(u_2,u_3;\psi_1);\psi_0) \\ &= \psi_0(\psi_0^{-1}(u_1) + \psi_0^{-1}(\psi_1(\psi_1^{-1}(u_2) + \psi_1^{-1}(u_3)))), \mathbf{u} \in \mathbb{I}^3. \end{aligned} \tag{4}$$

As in the case of ACs we can ask for necessary and sufficient condition for function C given by (3) to be a proper copula. Partial answer for this question in form of sufficient condition is contained in the following theorem [6].

Theorem 3. ***(McNeil (2009))****. If $\psi_j \in \Psi_\infty, j \in \{0, \ldots, d-2\}$ such that $\psi_k^{-1} \circ \psi_{k+1}$ have completely monotone derivatives for all $k \in \{0, \ldots, d-3\}$, then $C(\boldsymbol{u};\psi_0, \ldots, \psi_{d-2}), \boldsymbol{u} \in \mathbb{I}^d$, given by (3) is a copula.*

If we take the most simple 3-HAC given by (4), we can see that the condition for C to be a proper copula following from McNeil's theorem is $(\psi_0^{-1} \circ \psi_1)'$ to be completely monotone. As this condition will be essential for the rest of this paper we put it in individual definition.

Definition 6. Let $\psi_a, \psi_b \in \Psi_\infty, a, b \in \{0, \ldots, d-2\}, a \neq b$ and $C(\cdot;\psi_a)$ corresponds to parent of $C(\cdot;\psi_b)$ in the tree structure of C. Then condition for $(\psi_a^{-1} \circ \psi_b)'$ to be compete monotone is called *nesting condition*.

As we can observe, verification of conditions in McNeil's theorem is just $d-2$ verifications of nesting condition for $d-2$ different pairs $\psi_k, \psi_{k+1}, k \in \{0, \ldots, d-2\}$. McNeil's theorem is stated only for fully-nested HACs, but it can be easily translated also for use with partially-nested HACs.

For the sake of simplicity, assume that each d-HAC structure corresponds to some binary tree t. Each node in t represents one 2-AC. Each 2-AC is determined just by its corresponding generator, so we identify each node in t with one generator and hence we have always nodes $\psi_0, \ldots, \psi_{d-2}$. For a node ψ denote as $\mathcal{D}_n(\psi)$ the set of all descendant nodes of ψ, $\mathcal{P}(\psi)$ the parent node of ψ, $\mathcal{H}_l(\psi)$ the left child of ψ and $\mathcal{H}_r(\psi)$ the right child of ψ. The leafs of t correspond to the variables $u_1, \ldots, u_d$.

2.4 Measure of concordance

A measure of concordance (MoC) is a measure, which reflects a degree of dependency between two random variables independently on their univariate distributions. There also exist generalizations for more than two random variables, but we present only pairwise measure of concordance. As $\mathcal{C}$ allows for partial ordering known as *concordance ordering*, a measure of concordance also reflects this ordering (see [3, 7]). One of the most popular measures of concordance is *Kendall's tau*. As we are interested in its relationship with a general bivariate copula, we use its the definition given by (as in [1])

$$\tau(C) = 4 \int_{\mathbb{I}^2} C(u_1,u_2) dC(u_1,u_2) - 1. \tag{5}$$

If C is 2-AC based on a generator ψ and ψdepends on the parameter θ, then (5) states a relationship between θ and τ. This relationship is very important for our approach and is used extensively later in Section 3.

2.5 Okhrin's algorithm for the structure determination of HAC

We recall the algorithm presented in [8] for the structure determination of HAC, which returns for some unknown HAC C its structure using only the known forms of its bivariate margins. The algorithm uses the following definition.

Definition 7. Let C be a d-HAC with generators $\psi_0, \ldots, \psi_{d-2}$ and $(U_1, \ldots, U_d) \sim C$. Then denote as $\mathcal{U}_C(\psi_k), k = 0, \ldots, d-2$, the set of indexes $\mathcal{U}_C(\psi_k) = \{i | (\exists U_j)(U_i, U_j) \sim C(\cdot;\psi_k) \vee (U_j, U_i) \sim C(\cdot;\psi_k), 1 \leq i < j \leq d\}, k = 0, \ldots, d-2$.

Proposition 4. *Defining* $\mathcal{U}_C(u_i) = \{i\}$ *for the leaf* $i, 1 \le i \le d$, *there is an unique disjunctive decomposition of* $\mathcal{U}_C(\psi_k)$ *given by*

$$\mathcal{U}_C(\psi_k) = \mathcal{U}_C(\mathcal{H}_l(\psi_k)) \cup \mathcal{U}_C(\mathcal{H}_r(\psi_k)). \tag{6}$$

Due to space limitations we do not state the proof for the proposition and we refer the reader to the Okhrin's work [8], which includes detailed description of the method and the necessary proofs.

For an unknown d-HAC C, knowing all its bivariate margins, its structure can be easily determined with Algorithm 1, which returns the unknown structure t of C. We start from the sets $\mathcal{U}_C(u_1), ..., \mathcal{U}_C(u_d)$ joining them together through (6) until we reach the node ψ for which $\mathcal{U}_C(\psi) = \{1, ..., d\}$.

Algorithm 1 The HAC structure determination

$\mathcal{I} = \{0, ..., d-2\}$
while $\mathcal{I} \neq \emptyset$ **do**
1. $k = \mathrm{argmin}_{i \in \mathcal{I}}(\#\mathcal{U}_C(\psi_i))$, if there are more minima, then choose as k one of them arbitrarily.
2. Find the nodes ψ_l, ψ_r, for which $\mathcal{U}_C(\psi_k) = \mathcal{U}_C(\psi_l) \cup \mathcal{U}_C(\psi_r)$.
3. $\mathcal{H}_l(\psi_k) := \psi_l, \mathcal{H}_r(\psi_k) := \psi_r$.
4. Set $\mathcal{I} := \mathcal{I} \backslash \{k\}$.

end while

3 Our approach

Recalling Theorem 3, the sufficient condition for C to be a proper copula is, that the nesting condition must hold for each generator and its parent in a HAC structure. As this is the only known condition that assures that C is a proper copula, we concern in this work only the copulas, which fulfill this condition. The nesting condition results in constraints for the parameters θ_0, θ_1 of the involved generators ψ_0, ψ_1 (see [4, 3]). As $\theta_i, i = 1, 2$ is closely related to a MoC, e.g. τ and θ_i relationship established through (5), there is also an important relationship between the MoC and the HAC tree structure following from the nesting condition. This relationship is described for the fully-nested 3-HAC given by the form (4) in Remark 2.3.2 in [3]. There is stated that if the nesting condition holds for the parent-child pair (ψ_0, ψ_1), then $0 \le \kappa(\psi_0) \le \kappa(\psi_1)$, where κ is a MoC (as we concern only HACs with binary structures, which incorporates only 2-ACs, which are fully determined only by its generator, we use as domain of κ the set Ψ instead of the usually used set of all 2-copulas). We generalize this statement, using our notion, as follows.

Proposition 5. *Let* C *be a* d*-HAC with the structure* t *and the generators* $\psi_0, ..., \psi_{d-2}$, *where each parent-child pair satisfy the nesting condition. Let* κ *be a MoC. Than* $\kappa(\psi_i) \le \kappa(\psi_j)$, where $\psi_j \in \mathcal{D}_n(\psi_i)$, *holds for each* $\psi_i, i = 0, ..., d-2$.

Proof. If $\psi_i = \mathcal{P}(\psi_j)$, then we get directly $\kappa(\psi_i) \le \kappa(\psi_j)$ using Remark 2.3.2 from [3]. Otherwise, as $\psi_j \in \mathcal{D}(\psi_i)$, there exists a unique sequence $\psi_{k_1}, ..., \psi_{k_l}$, where $0 \le k_m \le d-2, m = 1, ..., l, l \le d-1, \psi_{k_1} = \psi_i, \psi_{k_l} = \psi_j$ and $\psi_{k-1} = \mathcal{P}(\psi_k)$ for $k = 2, ..., l$. Applying the above mentioned remark for each pair $(\psi_{k-1}, \psi_k), k = 2, ..., l$, we get $\kappa(\psi_{k_1}) \le ... \le \kappa(\psi_{k_l})$. □

Thus, having a branch from t, all its nodes are uniquely ordered according to their value of κ assuming unequal values of κ for all parent-child pairs. This provides us an alternative algorithm for the HAC structure determination. We have to assign the generators with the highest values of κ to the lowest levels of the branches in the structure and ascending to higher levels we assign the generators with lower values of κ.

To allow for computation of MoC among m (possibly > 2) random variables (r.v.s) we state the following definition. For simplification, denote the set of pairs of r.v.s as $\mathbf{U}_{IJ} = \{(U_i, U_j) | (i, j) \in I \times J\}$, where $I, J \subset \mathbb{N}, I \neq \emptyset \neq J$.

Definition 8. Let $m \in \mathbb{N}$ and κ be a MoC. Then define an *aggregated* MoC κ^+ as

$$\kappa^+(\mathbf{U}_{IJ}) = \begin{cases} \kappa(U_i, U_j) & \text{if } I = \{i\}, J = \{j\} \\ +(\kappa(U_i, U_j)_{i \in I, j \in J}), & \text{else,} \end{cases} \tag{7}$$

where the non-empty sets $I, J \subset \{1, ..., m\}, I \cap J = \emptyset$ and $+$ denotes an aggregation function[3], for which $+(x, ..., x) = x$ for all $x \in \mathbb{I}$.

Remark 3. $\kappa(\psi_k) = \kappa^+(\mathbf{U}_{\mathcal{U}_C(\mathcal{H}_l(\psi_k))\mathcal{U}_C(\mathcal{H}_r(\psi_k))})$ for a d-HAC C and for each $k = 0, ..., d-2$.

Let us illustrate our approach to the structure determination for $d = 4$. Assume three different structures t_1, t_2, t_3 corresponding to copulas C_1, C_2, C_3. For t_1 let $\mathcal{U}_{C_1}(\psi_2) = \mathcal{U}_{C_1}(\psi_1) \cup \mathcal{U}_{C_1}(\psi_0) = \{3, 4\} \cup \{1, 2\}$. For simplification denote $\{3, 4\} \cup \{1, 2\}$ as $((34)(12))$. For t_2 let $\mathcal{U}_{C_2}(\psi_2) = \{u_4\} \cup (\mathcal{U}_{C_2}(\psi_1) \cup \mathcal{U}_{C_2}(\psi_0)) = \{u_4\} \cup (\{u_3\} \cup \{u_1, u_2\}) = (4(3(21)))$. For t_3 let $\mathcal{U}_{C_3}(\psi_0) = (3(4(12)))$. We see that t_1 is the structure of a partially-nested 4-HAC and t_2, t_3 are the structures of fully-nested 4-HACs. Also assume (without a loss of generality) $\kappa(\psi_2) = \alpha, \kappa(\psi_0) = \gamma$ and $\alpha < \kappa(\psi_1) < \gamma, \alpha, \gamma \in \mathbb{I}$ for all t_1, t_2, t_3. The case when $\alpha = \kappa(\psi_1)$ or $\kappa(\psi_1) = \gamma$ is discussed later for a 3-HAC. Denote $\beta_1 = \kappa(\psi_1)$ for t_1, $\beta_2 = \kappa(\psi_1)$ for t_2 and $\beta_3 = \kappa(\psi_1)$ for t_3. The quantities $\alpha, \beta_1, \beta_2, \beta_3, \gamma$ can be determined from corresponding bivariate distributions as for t_1 is $\alpha = \kappa(\psi_2) = \kappa(U_3, U_1) = \kappa(U_3, U_2) = \kappa(U_4, U_1) = \kappa(U_4, U_2), \beta_1 = \kappa(\psi_1) = \kappa(U_3, U_4), \gamma = \kappa(\psi_0) = \kappa(U_1, U_2)$. For t_2 we have $\alpha = \kappa(\psi_2) = \kappa(U_4, U_3) = \kappa(U_4, U_1) = \kappa(U_4, U_2), \beta_2 = \kappa(\psi_1) = \kappa(U_3, U_1) = \kappa(U_3, U_2), \gamma = \kappa(\psi_0) = \kappa(U_1, U_2)$. For t_3 similarly $\alpha = \kappa(\psi_2) = \kappa(U_4, U_3) = \kappa(U_3, U_1) = \kappa(U_3, U_2), \beta_3 = \kappa(\psi_1) = \kappa(U_4, U_1) = \kappa(U_4, U_2), \gamma = \kappa(\psi_0) = \kappa(U_1, U_2)$.

Now assume a 4-HAC C with unknown structure $t \in \{t_1, t_2, t_3\}$ and $(U_1, U_2, U_3, U_4) \sim C$. Compute κ for all pairs of the r.v.s. If follows from the assumptions that $\kappa(U_1, U_2) = \gamma$ is always (for $t = t_1, t_2, t_3$) the maximum from those values. To satisfy Proposition 5, it is necessarily $\mathcal{U}_C(\psi_0) = \{12\}$, what assures through Algorithm 1 that ψ_0 is assigned to the lowest level of a branch from t. We introduce the a new variable $Z = (U_1, U_2)$, which represents r.v.s U_1, U_2. Once again compute κ for all the pairs of the new r.v.s, which are now r.v.s (U_3, U_4, Z). As Z represents two r.v.s we use generalized κ^+. Thus we get $\beta_1 = \kappa^+(U_3, U_4) = \kappa(U_3, U_4), \beta_2 = \kappa^+(U_3, Z) = \kappa^+(\mathbf{U}_{\{3\}\{12\}})$ and $\beta_3 = \kappa^+(U_4, Z) = \kappa^+(\mathbf{U}_{\{4\}\{12\}})$. Consider that under $t = t_1$ is $\beta_1 > \beta_2 = \beta_3 = \alpha$. Under $t = t_2$ is $\beta_2 > \beta_1 = \beta_3 = \alpha$ and under $t = t_3$ is $\beta_3 > \beta_1 = \beta_2 = \alpha$. The determination of $\mathcal{U}_C(\psi_1)$ in accordance with Proposition 5 is then obvious - $\mathcal{U}_C(\psi_1) = \{3, 4\}$ if $\beta_1 = \max(\beta_1, \beta_2, \beta_3)$ or $\mathcal{U}_C(\psi_1) = \{3, 2, 1\}$ if $\beta_2 = \max(\beta_1, \beta_2, \beta_3)$ or $\mathcal{U}_C(\psi_1) = \{4, 2, 1\}$ if $\beta_3 = \max(\beta_1, \beta_2, \beta_3)$. The set $\mathcal{U}_C(\psi_2) = \{4, 3, 2, 1\}$ for all t_1, t_2, t_3.

The described process is generalized in Algorithm 2 for arbitrary $d > 2$. The algorithm returns the sets $\mathcal{U}_C(z_{d+k+1})$ corresponding to the sets $\mathcal{U}_C(\psi_k), k = 0, ..., d-2$. Passing them to Algorithm 1, we avoid their computation from Definition 7 and we get the requested d-HAC structure without a need of knowing the forms of the bivariate margins.

Algorithm 2 The HAC structure determination based on κ

Input:

1) $\mathcal{I} = \{1, ..., d\}$, 2) $(U_1, ..., U_d) \sim C$, 3) κ^+ ... an aggregated MoC, 4) $z_k = u_k, \mathcal{U}_C(z_k) = \{k\}, k = 1, ..., d$

The structure determination:

for $k = 0, ..., d-2$ **do**

1. $(i, j) := \underset{i^* < j^*, i^* \in \mathcal{I}, j^* \in \mathcal{I}}{\operatorname{argmax}} \kappa^+(\mathbf{U}_{\mathcal{U}_C(z_{i^*})\mathcal{U}_C(z_{j^*})})$
2. $\mathcal{U}_C(z_{d+k+1}) := \mathcal{U}_C(z_i) \cup \mathcal{U}_C(z_j)$
3. $\mathcal{I} := \mathcal{I} \cup \{d + k + 1\} \backslash \{i, j\}$

end for

Output:

$\mathcal{U}_C(z_{d+k+1}), k = 0, ..., d-2$

Now consider a fully-nested 3-HAC with two equal generators given by the form $C(u_1, C(u_2, u_3; \psi); \psi) = \psi(\psi^{-1}(u_1) + \psi^{-1}(\psi(\psi^{-1}(u_2) + \psi^{-1}(u_3))))$. As it equals to $\psi(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \psi^{-1}(u_3))$, which is the 3-AC $C(u_1, u_2, u_3; \psi)$, we get for this copula in Step 1. of the algorithm three pairs (1,2), (1,3), (2,3) corresponding to the maximal value of κ^+. This is because all bivariate margins of $C(u_1, u_2, u_3; \psi)$ are distributed equally. Choosing the first pair to be the pair (i, j) we get the result of the algorithm as $\mathcal{U}_C(\psi_0) = \{1, 2\}, \mathcal{U}_C(\psi_1) = \{1, 2, 3\}$. Passing it to Algorithm 1 we get the corresponding structure and denote it as r_1. In the same way we obtain for the second and the third pair the structures we denote as r_2, r_3. But, as $C(u_1, C(u_2, u_3; \psi); \psi) = C(u_1, u_2, u_3; \psi)$, all those structures r_1, r_2, r_3 corresponds to the same copula. Thus, in the case that there are more than one pair corresponding to the maximal value of κ^+ in Step 1., we can choose the pair arbitrarily, because it does not affect the resulting copula, i.e.

[3] like, e.g., max, min or mean

the algorithm return different structures, which however correspond to the same copula. This fact can be also easily generalized for the case when $d > 3$.

4 Conclusions

As the aggregated κ^+ depends only on the pairwise κ and the aggregation function $+$, we can easily derive its empirical version κ_n^+ just by substituting κ in κ^+ by its empirical version κ_n, e.g., by empirical version of Kendall's tau. Using κ_n^+ instead of κ^+ we can easily derive the empirical version of the structure determination process represented by Algorithms 1, 2. Conclude that in this way we base the structure determination only on the values of the pairwise MoC. This is the essential property of our approach, because if the relationship between κ and θ established through (5) is explicitly known, whole HAC, including its structure and its parameters, can be estimated just from κ_n computed on the realizations of $(U_i, U_j), 1 \leq i < j \leq d$ completely avoiding the use of the MLE.

Acknowledgements

This work has been originally published in the proceedings of Mathematical Methods in Economics 2013, Jihlava, see `https://mme2013.vspj.cz/about-conference/conference-proceedings`. For a BibTex citation, see `http://suzelly.opf.slu.cz/~gorecki/en/research.php`.

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